Radiation of solitons described by a high-order cubic nonlinear Schrödinger equation

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The resonant radiation of solitons due to higher order dispersion, described by an extended nonlinear Schrödinger (NLS) equation with nonlinear (cubic) dispersive terms and linear terms with third and fourth derivatives, is studied. The basic equation includes, as a particular case, a higher order derivative NLS equation. General properties of the master equation, such as conservation laws, Hamiltonian structures (in important particular cases), and Galilei transformation are studied. Explicit asymptotic expressions, describing the radiation at different initial conditions, are derived. The obtained results, in particular, provide a basis for the study of soliton losses, caused by the radiation, in optical fibers.

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I. INTRODUCTION

We consider the resonant radiation of solitons described by highly dispersive cubic nonlinear Schrödinger (NLS) equation

$$i\partial_T\Psi+\frac{1}{2}\partial_X^2\Psi+\alpha|\Psi|^2\Psi+i\alpha_1|\Psi|^2\partial_X\Psi+i\alpha_2\Psi\partial_X|\Psi|^2$$
$$+i\alpha_3\partial_X^3\Psi+\alpha_4\partial_X^4\Psi=0, \tag{1}$$

where α and α_n, $1\leqslant n\leqslant 4$, are real constants. Equation (1) plays an important role in the nonlinear fiber optics [1]. The radiation of a NLS soliton caused by higher order linear dispersion with $\alpha_1=\alpha_2=0$ and the reverse action of radiation on a moving soliton were studied in Refs. [2,3]. In this paper, we shall apply Eq. (1) to the investigation of the soliton resonant radiation taking into account the nonlinear dispersive terms with nonvanishing α_1 and α_2. (The term α_2 describes induced Raman effect and, generally, α_2 should be complex. In many cases, however, $\mathrm{Im}\,\alpha_2 \ll \mathrm{Re}\,\alpha_2$ and in this paper $\mathrm{Im}\,\alpha_2$ is neglected.) Three particular cases of Eq. (1) are of special interest.

(i) At

$$\alpha=1,\ |\alpha_3|+|\alpha_4|\ll 1,\ |\alpha_1|+|\alpha_2|\ll 1 \tag{2a}$$

we shall find corrections to the results obtained in Refs. [2, 3], where nonlinear dispersive terms had been neglected.

(ii) In the case

$$\alpha=1,\ |\alpha_3|+|\alpha_4|\leqslant 1,\ |\alpha_1|+|\alpha_2|\leqslant 1,\ \text{or}\ |\alpha_1|+|\alpha_2|\sim 1, \tag{2b}$$

we find and investigate the radiation of solitons with account of cubic dispersive terms.

(iii) At

$$\alpha=0,\quad |\alpha_3|+|\alpha_4|\ll|\alpha_1|+|\alpha_2|\sim 1, \tag{2c}$$

we will investigate the influence of higher order dispersion on the soliton of derivative NLS (DNLS) equations [4,5]. [At certain relations between coefficients α_n, Eq. (1) is solvable by the inverse scattering transform [6,7]; however, these relations are very special.]

The radiation should lead to the soliton attenuation and corresponding variations of its parameters. For the soliton with sufficiently large lifetime, its parameters should change slowly enough. As in previous papers [2,3,8,9] (see also references therein) we assume that the soliton radiation is small enough and use an adiabatic approach, considering first the radiation of a soliton with constant parameters; then the changes of soliton parameters can be evaluated from conservation laws. It will be shown that the intensity of soliton radiation is exponentially small if the soliton width is much larger than the inverse wave number of radiation. Therefore the adiabatic approximation is well justified if the ratio of the soliton width to the inverse wave number of radiated wave is a large parameter (a necessary condition of that is a smallness of coefficients in linear highly dispersive terms). The soliton attenuation, caused by the radiation, will be studied in a forthcoming paper.

This paper is organized as follows. In Sec. II we consider the conservation laws that follow from Eq. (1). It appears that the forms of conserving quantities essentially depend on relations between the coefficients in Eq. (1). Two important particular cases are studied in this section: (1) $\alpha_2=0$ and (2) $\alpha=0$, $\alpha_1=\alpha_2$. Apart from the conservation laws, we describe the Hamiltonian structures for these cases and show that they are also rather different. The results of this section will be used in subsequent papers for the investigation of soliton attenuation caused by the radiation. In Sec. III we consider the Galilei transformation for Eq. (1) and apply it for the description of moving solitons. In Sec. IV we investigate the soliton solutions to Eq. (1) in ‘‘zero approximation,’’ neglecting the linear terms with (small) α_3 and α_4. The restrictions on α_1 and α_2 are not imposed, except for $\mathrm{Im}\,\alpha_2=0$. The results of this section are used in Sec. V, where the asymptotic theory of the resonant soliton radiation is developed. In Sec. VI we discuss and summarize the obtained results.

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II. CONSERVATION LAWS: HAMILTONIAN STRUCTURES

The simplest conserving quantity, following from Eq. (1), is the so-called ‘‘adiabatic invariant’’

$$N=\int_{-\infty}^{\infty}|\Psi(X,T)|^2\,dX,\quad \frac{dN}{dt}=0. \tag{3}$$

This follows from the equation

$$\partial_T|\Psi|^2+\partial_X\left(\frac{i}{2}(\Psi\partial_X\Psi^*-\Psi^*\partial_X\Psi)+\frac{1}{2}(\alpha_1+2\alpha_2)|\Psi|^4\right)$$
$$+\alpha_3\partial_X(\Psi^*\partial_X^2\Psi+\Psi\partial_X^2\Psi^*-|\partial_X\Psi|^2)-i\alpha_4\partial_X(\Psi^*\partial_X^3\Psi$$
$$+\partial_X^2\Psi^*\partial_X\Psi-\Psi\partial_X^3\Psi^*-\partial_X^2\Psi\partial_X\Psi^*)=0$$

which can be easily derived from Eq. (1). Other conserving quantities depend on relations between coefficients in Eq. (1). As important examples, we consider two cases.

(1) Case $\alpha_2=0$. In this case, Eq. (1) can be obtained from the variational principle with the Lagrange density

$$L=\frac{1}{2}i(\Psi^*\partial_T\Psi-\Psi\partial_T\Psi^*)-\frac{1}{2}|\partial_X\Psi|^2+\frac{1}{2}\alpha|\Psi|^4$$
$$+\frac{1}{4}i\alpha_1|\Psi|^2(\Psi^*\partial_X\Psi-\Psi\partial_X\Psi^*)$$
$$+\frac{1}{2}i\alpha_3(\Psi^*\partial_X^3\Psi-\Psi\partial_X^3\Psi^*)+\alpha_4|\partial_X^2\Psi|^2. \tag{4}$$

From Eq. (4) we find, in addition to Eq. (3), two more conservation laws:

$$\frac{dP}{dt}=0,\quad P=\frac{1}{2}i\int_{-\infty}^{\infty}(\Psi\partial_X\Psi^*-\Psi^*\partial_X\Psi)dX, \tag{5}$$

$$\frac{dH}{dt}=0,\quad H=\frac{1}{2}i\int_{-\infty}^{\infty}(\Psi^*\partial_T\Psi-\Psi\partial_T\Psi^*)dX-\int_{-\infty}^{\infty}L\,dX. \tag{6}$$

Expressions for P and H follow from the invariance of the Lagrangian with respect to space and time shifts. Therefore, P will be called momentum and H total energy of the wave field. Substituting L in Eq. (6), we arrive at the expression

$$H\{\Psi,\Psi^*\}=\int_{-\infty}^{\infty}dX\Bigg\{\frac{1}{2}|\partial_X\Psi|^2-\frac{1}{2}\alpha|\Psi|^4$$
$$-\frac{1}{4}i\alpha_1|\Psi|^2(\Psi^*\partial_X\Psi-\Psi\partial_X\Psi^*)$$
$$-\frac{1}{2}i\alpha_3(\Psi^*\partial_X^3\Psi-\Psi\partial_X^3\Psi^*)-\alpha_4|\partial_X^2\Psi|^2\Bigg\}. \tag{7}$$

One can check that Eq. (1) can be expressed through the variational derivative of the functional $H\{\Psi,\Psi^*\}$

$$i\partial_t\Psi=\frac{\delta H\{\Psi,\Psi^*\}}{\delta\Psi^*}. \tag{8}$$

Equation (8) is a particular case of a general equation for any functional $A\{\Psi,\Psi^*\}$

$$\partial_t A=[H,A], \tag{9}$$

where $[A\{\Psi,\Psi^*\},B\{\Psi,\Psi^*\}]$ is the Poisson bracket, defined as for the regular NLS equation [10]

$$[A\{\Psi,\Psi^*\},B\{\Psi,\Psi^*\}]$$
$$=i\int_{-\infty}^{\infty}dX\left(\frac{\delta A}{\delta\Psi}\frac{\delta B}{\delta\Psi^*}\frac{\delta B}{\delta\Psi^*}-\frac{\delta A}{\delta\Psi^*}\frac{\delta B}{\delta\Psi}\right), \tag{10}$$

and therefore ensuring the validity of the Jakobi identity. Thus the wave field governed by Eq. (1) with $\alpha_2=0$ is a Hamiltonian system [11].

(2) Case $\alpha=0$, $\alpha_1=\alpha_2$. Here, Eq. (1) can be written as

$$i\partial_T\Psi+\frac{1}{2}\partial_X^2\Psi+i\alpha_1\partial_X(|\Psi|^2\Psi)+i\alpha_3\partial_X^3\Psi+\alpha_4\partial_X^4\Psi=0, \tag{11}$$

and one can define the Poisson brackets as for the DNLS equation [4]

$$[A\{\Psi,\Psi^*\},B\{\Psi,\Psi^*\}]$$
$$=\int_{-\infty}^{\infty}dX\left(\frac{\delta B}{\delta\Psi}\partial_X\frac{\delta A}{\delta\Psi^*}+\frac{\delta B}{\delta\Psi^*}\partial_X\frac{\delta A}{\delta\Psi}\right). \tag{12}$$

Defining the Hamiltonian as

$$H=\int_{-\infty}^{\infty}dX\Bigg(\frac{1}{2}i\Psi^*\partial_X\Psi-\frac{1}{2}\alpha_1(\Psi\Psi^*)^2$$
$$-\alpha_3\Psi^*\partial_X^2\Psi+i\alpha_4\Psi^*\partial_X^3\Psi\Bigg), \tag{13}$$

one can check that equation

$$\partial_t\Psi=[H,\Psi] \tag{14}$$

leads to Eq. (1) with $\alpha=0$, $\alpha_1=\alpha_2$, and the evolution of any functional $A\{\Psi,\Psi^*\}$ is determined by Eqs. (9) and (12). Evidently, for the case in question one can again check the conservation of the adiabatic invariant $N\{\Psi,\Psi^*\}$ from Eqs. (3), and $H\{\Psi,\Psi^*\}$ from Eq. (13).

III. MODIFIED GALILEI TRANSFORMATION

The transformation

$$\Psi(X,T)=\psi(X-VT,T)e^{i(KX-\Omega T)}, \tag{15}$$

being applied to the regular NLS equation ($\alpha_n=0$) with

$$K=V,\quad \Omega=\frac{1}{2}K^2, \tag{16}$$

describes the transition to the reference frame moving with velocity V and leaves the regular NLS equation unchanged. This well known result can be expressed as the invariance of the regular NLS equation with respect to the Galilei transformation (15),(16). Applying ansatz (15) to Eq. (1), we come to the equation of the same structure if

$$\Omega=\frac{1}{2}K^2-\alpha_3K^3-\alpha_4K^4, \tag{17}$$

$$V=K-3\alpha_3K^2-4\alpha_4K^3. \tag{18}$$

Indeed, the transformation (15) with Eqs. (17) and (18), determining Ω and K at given V, leads to the following equation for ψ

$$i\partial_t\psi(x,t)+\frac{1}{2}a_2\partial_x^2\psi+q|\psi|^2\psi+i\alpha_1|\psi|^2\partial_x\psi+i\alpha_2\psi\partial_x|\psi|^2+ia_3\partial_x^3\psi+a_4\partial_x^4\psi=0, \tag{19}$$

where

$$x=X-VT, \quad t=T, \tag{20}$$

$$a_2=1-6\alpha_3K-12\alpha_4K^2, \tag{21}$$

$$a_3=\alpha_3+4\alpha_4K, \quad a_4=\alpha_4, \tag{22}$$

$$q=\alpha-\alpha_1K. \tag{23}$$

(In the following we assume $a_2>0$.) Equations (1) and (19), in spite of different coefficients, have a similar structure. This allows us to say that Eq. (1) is *essentially* invariant with respect to the Galilei transformation. At $\alpha_n=0$, we have the complete Galilei invariance.

IV. SOLITON SOLUTIONS OF EQ. (1) WITHOUT LINEAR HIGHLY DISPERSIVE TERMS

Consider soliton solutions to Eq. (1) at

$$\alpha_3=\alpha_4=0. \tag{24}$$

They were derived in a number of papers; we shall do this in a form convenient for the investigation of the soliton radiation. Looking for the soliton solution in the form

$$\psi_s(x,t)=\exp\left\{i\left[\frac{1}{2}\lambda^2t+\varphi(x)\right]\right\}u(x), \qquad u(x)\to 0 \quad (x\to\pm\infty), \tag{25}$$

we substitute Eqs. (25) and (24) into Eq. (19) at (24). After some calculations (see the Appendix), we arrive at the equations

$$(u')^2=\lambda^2u^2-qu^4-A^2u^6, \tag{26}$$

$$\varphi'=-\frac{1}{2}(\alpha_1+2\alpha_2)u^2, \tag{27}$$

where

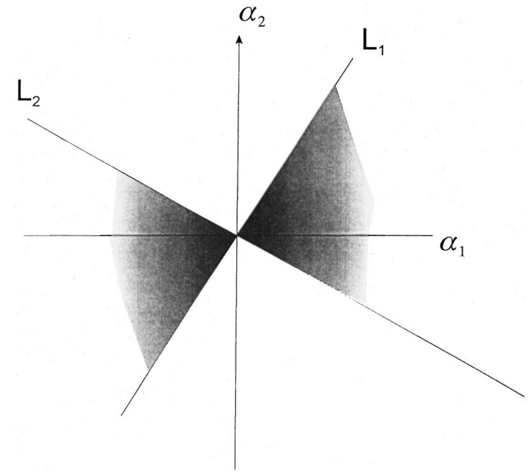

FIG. 1. Domain in the plane (α_1,α_2) where $A^2>0$ (hatched). It is situated between the lines L_1 $[\alpha_2=(3/2)\alpha_1]$ and L_2 $[\alpha_2=-(1/2)\alpha_1]$.

$$u'=\frac{du}{dx}, \quad \varphi'=\frac{d\varphi}{dx}, \tag{28}$$

and

$$A^2=\frac{4\alpha_1(\alpha_1+2\alpha_2)-(\alpha_1+2\alpha_2)^2}{12}. \tag{29}$$

These results are approximately valid at sufficiently small $\alpha_{3,4}$ if K, connected with the soliton velocity, satisfies limitations following from (21):

$$6\alpha_3K\ll 1, \quad 12\alpha_4K^2\ll 1. \tag{30}$$

Equation (26) can be written as

$$(u')^2=A^2u^2(u_0^2-u^2)(u_1^2+u^2), \tag{31}$$

where

$$u_0^2=\frac{\sqrt{4A^2\lambda^2+q^2}-q}{2A^2}, \tag{32}$$

$$u_1^2=\frac{\sqrt{4A^2\lambda^2+q^2}+q}{2A^2}. \tag{33}$$

It is easy to see that

$$u_1^2\geqslant u_0^2>0 \quad (q\geqslant 0), \quad u_0^2\geqslant u_1^2>0 \quad (q\leqslant 0), \tag{34}$$

provided

$$A^2>0. \tag{35}$$

Condition (35), which will be assumed hereafter, imposes restrictions on α_1 and α_2: it is satisfied in the region of the plane (α_1,α_2), shown in Fig. 1. In particular, this region includes cases $\alpha_2=0$ and $\alpha_1=\alpha_2$, considered in Sec. II; this agrees with the explicit expression

$$A^2=\frac{\alpha_1^2}{4\alpha_2} \quad (\alpha_2=0 \ \text{ or } \ \alpha_1=\alpha_2) \tag{36}$$

that follows from Eq. (29). [If condition (35) is satisfied, we define $A>0$.]

From Eqs. (35) and (31) it is seen that in the soliton

$$0<u(x)\leqslant u_0$$

and, therefore, u_0 is the soliton amplitude. From expressions (32) and (33) one obtains

$$u_1=\lambda/Au_0. \tag{37}$$

It can also be seen from Eq. (26), that the soliton width is of order λ^{-1}. Indeed, at large x, where $u(x)$ is small, we have

$$u'\approx\pm\lambda u,\quad \lambda>0. \tag{38}$$

Therefore, the inverse width of the soliton is of order λ. Equation (37) expresses u_1 through the soliton width, amplitude, and parameter $A(\alpha_1,\alpha_2)$.

Integrating Eqs. (26) and (27), we have

$$u(x)=u_0u_1[u_0^2\sinh^2(\lambda x)+u_1^2\cosh^2(\lambda x)]^{-1/2}, \tag{39}$$

$$\varphi(x)=-\frac{\alpha_1+2\alpha_2}{2A}\arctan\left[\frac{u_0}{u_1}\tanh(\lambda x)\right]. \tag{40}$$

The constant of integration in Eq. (40) is insignificant.

These expressions approach the soliton solution of the regular NLS equation (where $\alpha=1$) if

$$4A^2\lambda^2\ll q^2, \tag{41}$$

$$q\approx 1\quad (\text{i.e.,}|\alpha_1K|\ll 1). \tag{42}$$

Indeed, from Eqs. (32),(33) and (41),(42) it follows

$$u_0\approx\lambda,\quad u_1\approx 1/A,\quad (u_0/u_1)^2\approx A^2\lambda^2\ll 1. \tag{43}$$

Then Eqs. (39) and (40) are approximately reduced to the soliton solution of the regular NLS equation:

$$u(x)\approx u_0\,\mathrm{sech}(u_0x),\quad \varphi(x)\approx 0. \tag{44}$$

V. RESONANT SOLITON RADIATION

A. Quasisolitons and resonant wave numbers

At $|\alpha_3|+|\alpha_4|>0$, Eqs. (1) and (19) respectively, may not have soliton solutions vanishing at $|x|\to\infty$ and regular in the limit $|\alpha_3|+|\alpha_4|\to 0$. The corresponding criterion at

$$|\alpha_3|+|\alpha_4|\ll 1, \tag{45}$$

will be obtained below. However, even if such solitons do not exist but condition (45) is held, one can find weakly radiating (and therefore slowly attenuating) pulses looking like solitons approximately satisfying Eq. (1). They can be called quasisolitons and will be studied in this section, together with their radiation. Our approach is similar to the one developed in Refs. [2,3] where quasisolitons, their radiation, and evolution were investigated for $\alpha_{1,2}=0$ (see also references cited in Refs. [1,2]).

Assuming Eq. (45), we write the solution of Eq. (19) in the form

$$\psi(x,t)=[\tilde{u}(x)+\eta(x,t)]\exp\left[\frac{1}{2}i\lambda^2t+i\tilde{\varphi}(x)\right], \tag{46}$$

where

$$\tilde{\psi}_s(x,t)=\tilde{u}(x)\exp\left[\frac{1}{2}i\lambda^2t'+i\tilde{\varphi}(x)\right], \tag{47}$$

is the soliton solution of Eq. (19) without the two last terms.

Expressions for $\tilde{u}(x)$ and $\tilde{\varphi}(x)$ can be found similar to Eqs. (39) and (40); the only difference is that now $a_2\neq 1$. It is easy to find that

$$\tilde{u}(x)=\tilde{u}_0\tilde{u}_1\left[\tilde{u}_0^2\sinh^2\left(\frac{\lambda}{\sqrt{a_2}}x\right)+\tilde{u}_1^2\cosh^2\left(\frac{\lambda}{\sqrt{a_2}}x\right)\right]^{-1/2}, \tag{48}$$

$$\tilde{\varphi}(x)=-\frac{\alpha_1+2\alpha_2}{2A}\arctan\left[\frac{\tilde{u}_0}{\tilde{u}_1}\tanh\left(\frac{\lambda}{\sqrt{a_2}}x\right)\right], \tag{49}$$

where a_2 is defined in Eq. (21) with $|\alpha_3|+|\alpha_4|\neq 0$ and $\tilde{u}_0,\tilde{u}_1$ are given by

$$\tilde{u}_{0,1}^2=\frac{a_2}{2A^2}\left(\sqrt{\frac{4A^2\lambda^2}{a_2}+q^2}\mp q\right). \tag{50a}$$

In the following we assume that $a_2>0$ and $a_2\sim 1$. It is easy to see that $\tilde{u}_0=\tilde{u}(0)$ is the amplitude of pulse (48) and, similar to Eq. (37):

$$\tilde{u}_1=\lambda\sqrt{a_2}/A\tilde{u}_0. \tag{50b}$$

Another term in the first bracket of Eq. (46), $\eta(x,t)$, is a small addition describing the influence of two last terms in Eq. (19). The equation for $\eta(x,t)$ can be obtained by substituting Eq. (46) into (19). It looks rather tedious, even after linearization with respect to $\eta(x,t)$. However, at $|x|\gg\lambda$, where the cross terms containing the products of $\tilde{u}(x)$ and $\eta(x,t)$ can be neglected, the resulting equation becomes much simpler and has the form

$$i\partial_t\eta+\frac{1}{2}a_2\partial_x^2\eta+ia_3\partial_x^3\eta+a_4\partial_x^4\eta-\frac{\lambda^2}{2}\eta=0. \tag{51}$$

It describes the asymptotic behavior of $\eta(x,t)$ at large $|x|$ and can be easily obtained directly from the linearized Eq. (19).

Looking for the solution to Eq. (51) in the form

$$\eta\propto\exp(ikx-i\omega t), \tag{52}$$

we come to the equation

$$2\omega-a_2k^2+2a_3k^3+2a_4k^4-\lambda^2=0. \tag{53}$$

At $\omega=0$ the wave (52) is steady in the soliton frame and therefore it resonantly interacts with the soliton, if k is real. From Eq. (53) we have the following equation for k at $\omega=0$

$$2a_4k^4+2a_3k^3-a_2k^2-\lambda^2=0. \tag{54}$$

Assume that $4\alpha_4 K \ll 1$. Then from Eqs. (45) and (22) it follows that

$$a_3 \ll 1, \quad a_4 \ll 1, \tag{55}$$

and Eq. (54) can be solved approximately. Neglecting the first two terms in Eq. (54), we have the two smallest roots of this equation

$$k_{3,4} \approx \pm i\lambda/\sqrt{a_2}. \tag{56}$$

They are purely imaginary; other two roots should be much larger. Neglecting the last term in Eq. (54), we arrive at the equation

$$2a_4k^2 + 2a_3k - a_2 = 0, \tag{57}$$

which gives two other roots

$$k_{1,2} \approx \frac{-a_3 \pm \sqrt{a_3^2 + 2a_2a_4}}{2a_4}. \tag{58}$$

Unlike $k_{3,4}$, they are real if

$$a_3^2 + 2a_2a_4 > 0, \tag{59}$$

(at $a_4 \geq 0$, this condition is automatically fulfilled).

To explore the meaning of the roots $k_{3,4}$, we substitute $\omega = 0$ and Eq. (56) into (52) and compare the result with Eq. (48). Then we see that expression (56) represents the inverse length of the soliton; this is a natural result because η, as a steady asymptotic solution in the soliton frame, must contain a term describing the soliton asymptotic behavior.

On the other hand, at condition (59), $k_{1,2}$ are the wave numbers of periodic waves resonantly interacting with the soliton. It will be seen below that such waves are emitted by the soliton and thus they may be called the resonant soliton radiation (RSR). Respectively, the roots [Eq. (58)] will be called resonant wave numbers. Condition (59) is, therefore, a necessary condition that the RSR does exist and Eq. (46) describes the quasisoliton and its radiation. For brevity we call the pulse (47)–(49) *radiating soliton* [though it is, in fact, not a soliton solution of full Eq. (19)].

If condition (59) does not hold and therefore all k_j's are complex, Eqs. (1) and (19) have regular soliton solutions. They can be found, in particular, by means of the perturbation approach, using Eqs. (47)–(50) as the zero approximation (the derivation of exact solutions requires rather tedious algebra). In this paper, however, we will investigate the case when condition (59) does hold and so we will discuss only the radiating solitons and their radiation.

The phase velocities of both branches of RSR are, evidently, equal to the velocity of radiating soliton. Their group velocities U_j $(j=1,2)$ can be calculated from Eq. (53). Differentiating Eq. (53) with respect to k, we have

$$U_j = \left(\frac{d\omega}{dk}\right)_{k=k_j} = a_2k_j - 3a_3k_j^2 - 4a_4k_j^3, \quad (j=1,2). \tag{60}$$

Using Eq. (57), we can exclude the term with a_4 to obtain

$$U_j \approx a_3k_j^2 - a_2k_j. \tag{61}$$

It is convenient to define $k_{1,2}$ in such a way that

$$|k_1| \leq |k_2|. \tag{62}$$

Now consider two limit cases. At

$$a_3^2 \gg 2a_2|a_4|, \tag{63}$$

we get

$$k_1 \approx \frac{a_2}{2a_3}, \quad k_2 \approx -\frac{a_3}{a_4}, \quad |k_1| \ll |k_2|, \tag{64}$$

$$U_1 \approx -\frac{a_2^2}{4a_3}, \quad U_2 \approx \frac{a_3^3}{a_4^2}. \tag{65}$$

In the opposite case

$$a_3^2 \ll 2a_2|a_4|, \tag{66}$$

condition (59) is fulfilled only at $a_4 > 0$ and we have

$$k_1 \approx -k_2 \approx -\left(\frac{a_2}{2a_4}\right)^{1/2}, \tag{67}$$

$$U_1 \approx -U_2 \approx a_2\left(\frac{a_2}{2a_4}\right)^{1/2}. \tag{68}$$

The results in Eqs. (51)–(68), naturally, coincide with those obtained in Ref. [2], where the cubic dispersive terms have not been considered. This is because they follow from the linearized Eq. (1).

B. Amplitudes and asymptotic behavior of the RSR

To get a complete description of RSR, one must solve Eq. (19) with proper initial conditions. For example, the condition

$$\eta(x,t) = 0$$

at

$$t = 0 \tag{69}$$

means that initially we assume a ''bare'' soliton pulse described by Eqs. (47)–(50). At $t>0$ the function $\eta(x,t)$ describes a soliton modification and the full radiation, consisting of the resonant radiation and a transient wave emitted due to the initial condition (69). As we have already mentioned, the full equation for $\eta(x,t)$ is rather complicated and it is, generally, difficult to find its analytical solution describing these effects. {Such a solution, however, has been found, e.g., for Langmuir solitons [12] where the equation for $\eta(x,t)$ is simpler. The soliton deformation and the full radiation can be seen in the numerical solutions of the original Eq. (1), as was demonstrated for the case $\alpha_1 = \alpha_2 = 0$ [3].} In this paper we will not study the soliton modification which is, generally, small at condition (45), as well as the transient effects that can be neglected at sufficiently large t and x. We shall confine ourself to the derivation of an asymptotic analytical expression describing the RSR at large t and x, using the approach developed earlier for some other systems in Refs. [8,9].

It is convenient, first, to consider instead of Eq. (69) the following initial condition

$$\eta(x,-t_0)=0, \quad t_0 \gg t_{\rm tr}, \tag{70}$$

where $t_{\rm tr}$ is the ''transient'' time, characterizing a duration of the transient effects. This permits us to suppose that at $t=0$ the soliton can be considered as ''dressed'' and described by

$$\psi_{\rm ds}(x)=W(x)\exp\left[i\Phi(x)+\frac{1}{2}i\lambda^2 t\right]. \tag{71}$$

The dressed soliton can be obtained, in principle, to any integer order of α_3 and α_4 using a perturbation theory (see, e.g., Refs. [1,13]). In the zero approximation, it is equal to the bare soliton ψ_s, described by Eqs. (47) and (48)–(50). Evidently,

$$\psi_{\rm ds}(x)\to 0, \quad (|x|\to\infty), \tag{72}$$

and the effective width of $\psi_{\rm ds}(x)$ is of the order of bare soliton width $\sqrt{a_2}/\lambda$; so $\psi_{\rm ds}(x)$ is exponentially small at $|x|\gg\sqrt{a_2}/\lambda$. Define

$$\frac{d}{dx}\ln\frac{\psi_{\rm ds}(x)}{\psi_{\rm ds}(0)}=i\left[\frac{d\Phi}{dx}+R(x)\right], \tag{73}$$

where

$$R(x)\equiv -i\frac{d}{dx}\ln\frac{W(x)}{W(0)}. \tag{74}$$

Then

$$W(x)=W(0)\exp\left[i\int_0^x R(\chi)d\chi\right]. \tag{75}$$

As far as the transient radiation is neglected and the part of $\eta(x,t)$ describing the soliton deformation is included in Eq. (71), we can write Eq. (46) at $t>0$ as

$$\psi(x,t)=\left\{W(0)\exp\left[i\int_0^x R(\chi)d\chi\right]+f(x,t)\right\}\times\exp\left[i\Phi(x)+\frac{1}{2}i\lambda^2 t\right], \tag{76}$$

where $f(x,t)$ is part of $\eta(x,t)$ describing the RSR. At sufficiently large t_0 in Eq. (70), we can temporally take $t_0=\infty$ and write the asymptotic expression

$$f=c_1\exp\left[i\int_0^x q_1(\chi)d\chi\right]\Theta(U_1x)+c_2\exp\left[i\int_0^x q_2(\chi)d\chi\right]\Theta(U_2x). \tag{77}$$

Here $q_{1,2}(x)$ are the wave numbers of RSR in the WKB approximation and $c_{1,2}$ are the amplitudes of the corresponding modes. Evidently,

$$q_1(x)\to k_1, \quad q_2(x)\to k_2, \quad (|x|\gg\sqrt{a_2}/\lambda),$$

where k_1,k_2 are the resonant wave numbers approximately given by Eq. (58) and $\Theta(Y)$ is the step function, defined as

$$\Theta(Y)=1 \ \ (Y>0), \quad \Theta(Y)=0 \quad (Y<0). \tag{78}$$

The Θ functions in Eq. (77) express that the directions of propagation and the localization of RSR are determined by the signs of the group velocities U_j $(j=1,2)$ given by Eq. (60). At sufficiently large $|x|/\lambda$, the first term in Eq. (76) can be neglected and then

$$\psi(x,t)\approx f(x)\exp\left[i\Phi(x)+\frac{1}{2}i\lambda^2 t\right]. \tag{79a}$$

On the other hand, at $|x|<\sqrt{a_2}/\lambda$, the term $f(x)$ is neglectable because $c_{1,2}$, as will be shown below, are very small with respect to the soliton amplitude. Thus, at $|x|<\sqrt{a_2}/\lambda$,

$$\psi(x,t)\approx W(0)\exp\left[i\int_0^x R(\chi)d\chi\right]\exp\left[i\Phi(x)+\frac{1}{2}i\lambda^2 t\right]. \tag{79b}$$

Following Refs. [8,9], we consider the continuations of $R(x)$ and $q_1(x),q_2(x)$ in the complex plane and assume that they are three branches of one analytic function. The branches strongly couple near the branch points that are the roots of equations

$$R(z_1)=q_1(z_1), \quad R(z_2)=q_2(z_2). \tag{80}$$

Generally, each of these equations has many roots in both half-planes and will consider the roots with the smallest imaginary parts which, as will be seen later, are the most essential. A pass around the branch point z_1 or z_2 leads to the transition $q_1(z)\to R(z)$ or $q_2(z)\to R(z)$.

To find the amplitudes c_1 and c_2 of RSR, we consider contours in the complex plane with the ends on the real axis, beginning at sufficiently large distance from the soliton $(|x|\gg\lambda)$. The branch points should be between the contours and the real axis (Fig. 2). [In the figure, $z_{1,2}$ is purely imaginary, which follows from symmetry considerations and can also be seen from the subsequently obtained solutions of Eq. (80).] The choice of half-planes is determined by the rule

$$\operatorname{Im} z_1=-\operatorname{sgn} k_1, \quad \operatorname{Im} z_2=-\operatorname{sgn} k_2. \tag{81}$$

Now consider, for example, a case $d_3<0$, $d_4>0$. Then

$$k_1<0, \quad U_1>0, \quad k_2>0, \quad U_2<0, \tag{82}$$

and $\operatorname{Im} z_1>0$, $\operatorname{Im} z_2<0$. Consequently, if one moves along the contours $C_{1,2}$ (Fig. 2), starting from the points $x_{1,2}$, the integrals

$$i\int_0^z q_{1,2}(\chi)d\chi\approx ik_{1,2}z,$$

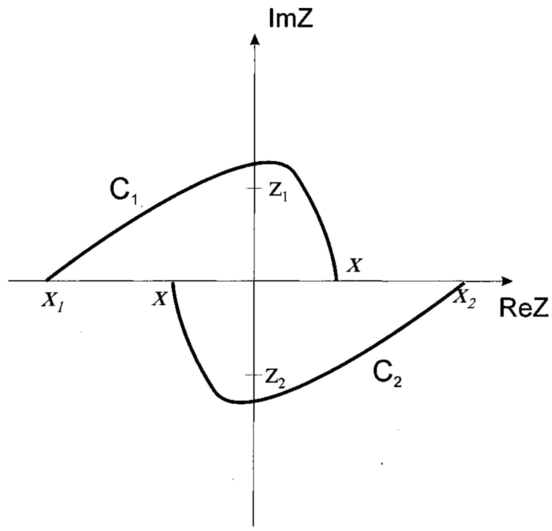

FIG. 2. Contours C_1 and C_2 in the complex plane for the case (82).

increase with $|\mathrm{Im}\, z|$. (Compare with a similar situation in the WKB theory [14].)

To calculate c_1, consider expression (76) after a trip along the contour C_1 with $x_1<0$ and $x>0$. Properly deforming C_1 we see that this is equivalent to a walk around the branch point z_1, which leads to the transformation

$$R(z)\to q_1(z).$$

On the other hand, traveling along the real axis x we have no such changes. From Ew. (76) it is seen that $\psi(x,t)$ will be the same in both cases if after the trip along the line C_1

$$W(x)\leftrightarrow f(x)=c_1\exp\left[i\int_0^x q_1(\chi)d\chi\right]. \tag{83}$$

On the other hand, from Eqs. (75) and (74) it follows that

$$W(x)=W(z_l)\exp\left[i\int_{z_1}^x R(\chi)d\chi\right]. \tag{84}$$

Then, after the pass along C_1

$$W(x)\to W(z_1)\exp\left[i\int_{z_1}^x q_1(\chi)d\chi\right]. \tag{85}$$

Comparing (83) and (85) we have

$$W(z_1)\exp\left[i\int_{z_1}^x q_1(\chi)d\chi\right]=c_1\exp\left[i\int_0^x q_i(\chi)d\chi\right],$$

which gives

$$c_1=W(z_1)\exp\left[-i\int_0^{z_1} q_1(\chi)d\chi\right]. \tag{86}$$

Analogously, traveling along the contour C_2, we get

$$c_2=W(z_2)\exp\left[-i\int_0^{z_2} q_2(\chi)d\chi\right]. \tag{87}$$

Equations (86) and (87) are similar to those obtained previously for other systems with high-order dispersion terms [9]. They seem to be of general significance. Their applications, however, become effective if one can use sufficiently simple approximate expressions of $W(z)$ and $q_{1,2}(z)$. Below we use rough approximations replacing

$$W(z)\to\tilde{u}(z),\quad q_{1,2}(z)\to k_{1,2}, \tag{88}$$

where $\tilde{u}(z)$ is given by Eq. (48), describing the bare soliton, and $k_{1,2}$ are the asymptotic resonant wave numbers of the RSR. Similar approximations were used in Refs. [8,9], where it was demonstrated that they are rather efficient, leading to correct results. Rigorous justification of Eq. (88) for complex z requires tedious algebra and will not be considered in this paper.

Thus, instead of Eqs. (86) and (87), we approximately write

$$c_{1,2}\approx\tilde{u}(z_{1,2})\exp(-ik_{1,2}z_{1,2}). \tag{89}$$

To find $z_{1,2}$, we replace Eq. (80) by approximate equations

$$R_0(z_1)\approx k_1,\quad R_0(z_2)\approx k_2, \tag{90}$$

where

$$R_0(z)=-i\frac{d}{dz}\ln\frac{\tilde{u}(z)}{\tilde{u}(0)}. \tag{91}$$

To simplify further calculations, it is convenient to transform Eq. (48) to the form

$$\tilde{u}(z)=\tilde{u}_0\frac{\sqrt{1+p}}{\sqrt{\cosh\left(\frac{2\lambda}{\sqrt{a_2}}z\right)+p}}, \tag{92a}$$

where

$$p=\frac{\tilde{u}_1^2-\tilde{u}_0^2}{\tilde{u}_1^2+\tilde{u}_0^2}=\frac{\sqrt{a_2}q}{\sqrt{4A^2\lambda^2+a_2q^2}}. \tag{92b}$$

From Eqs. (50a) and (92b) we have

$$\tilde{u}_0=\lambda\sqrt{\frac{2p}{(1+p)q}}. \tag{93}$$

From Eqs. (91) and (92a):

$$R_0(z)=\frac{\lambda}{\sqrt{a_2}}\frac{i\sinh\left(\frac{2\lambda}{\sqrt{a_2}}z\right)}{\cosh\left(\frac{2\lambda}{\sqrt{a_2}z}\right)+p}. \tag{94}$$

As far as wave numbers $k_{1,2}$ are large, the roots of Eq. (90) are located near the poles of $R_0(z)$. The poles with the smallest imaginary parts are

$$z_\pm=\pm i\frac{\sqrt{a_2}}{2\lambda}\left(\frac{\pi}{2}+\arcsin p\right). \tag{95}$$

Therefore, we look for the roots of Eq. (90) in the form

$$z_{1,2}=-i\,\mathrm{sgn}(k_{1,2})\frac{\sqrt{a_2}}{2\lambda}\left(\frac{\pi}{2}+\arcsin p-\zeta_{1,2}\right), \tag{96}$$

where

$$|\zeta_{1,2}|\leqslant\frac{\pi}{2}+\arcsin p. \tag{97}$$

[In Eq. (96) we also used Eq. (81).] From Eqs. (90), (94), and (96) we have the following equation for $\zeta_{1,2}$

$$\frac{p(1-\cos\zeta_{1,2})+\sqrt{1-p^2}\sin\zeta_{1,2}}{\sqrt{1-p^2}\cos\zeta_{1,2}+p\sin\zeta_{1,2}}=\frac{\lambda}{\sqrt{a_2}|k_{1,2}|}. \tag{98}$$

We will solve this equation assuming that its right-hand side is small, i.e.,

$$|k_{1,2}|\gg\lambda/\sqrt{a_2}. \tag{99}$$

This means that resonant wave numbers should be much larger than the inverse width of the soliton. (As will be seen from further results, only in this case is the intensity of RSR sufficiently small.) Due to Eq. (99), Eq. (98) has small roots $\zeta_{1,2}$ and can be written approximately as

$$\zeta_{1,2}\frac{p\zeta_{1,2}+2\sqrt{1-p^2}}{p\zeta_{1,2}+\sqrt{1-p^2}}=\frac{2\lambda}{\sqrt{a_2}|k_{1,2}|}. \tag{100}$$

Taking into account Eq. (97), we now consider two cases. The first one is

$$\min\left(\frac{\pi}{2}+\arcsin p,1\right)\gg|\zeta_{1,2}|\gg\sqrt{1-p^2}. \tag{101}$$

This is possible if p^2 is sufficiently close to one. If $p\approx-1$, the branch points (96) are close to the real axis and further results will show that in this case the RSR is rather strong. Then the losses of the soliton would be large, which is beyond our adiabatic approximation. Therefore this case will not be considered. So, we assume in Eq. (101) that p is close to one. Then,

$$\zeta_{1,2}\approx\frac{2\lambda}{\sqrt{a_2}|k_{1,2}|}\quad(1-p\ll1). \tag{102}$$

In this case, it follows from Eqs. (92b) and (93)

$$4A^2\lambda^2\ll q^2,\quad \tilde{u}_0\approx\lambda/\sqrt{q}. \tag{103}$$

If in addition $q\approx1$, we can neglect the nonlinear dispersive terms in Eq. (1), which leads to the theory at $\alpha_1=\alpha_2=0$ [2,3].

The second limit case is

$$|\zeta_{1,2}|\ll\sqrt{1-p^2}, \tag{104}$$

which includes, in particular, small p. Then

$$\zeta_{1,2}\approx\frac{\lambda}{\sqrt{a_2}|k_{1,2}|}. \tag{105}$$

Now we can calculate the amplitudes of RSR. According to Eq. (89), we must first calculate $\tilde{u}(z_{1,2})$. Using Eqs. (92), (96), and (97) we arrive at

$$\tilde{u}(z_{1,2})\approx\tilde{u}_0\left[\frac{2(1+p)}{2\zeta_{1,2}\sqrt{1-p^2}+p\zeta_{1,2}^2}\right]^{1/2}. \tag{106}$$

In case (101), using (102), we obtain

$$\tilde{u}(z_{1,2})\approx\frac{2\tilde{u}_0}{\zeta_{1,2}}\approx\tilde{u}_0\frac{\sqrt{a_2}|k_{1,2}|}{\lambda}. \tag{107}$$

In case (104), using (105), we have

$$\tilde{u}(z_{1,2})\approx\sqrt{\frac{\tilde{u}_0\tilde{u}_1}{\zeta_{1,2}}}\approx\left(\frac{\sqrt{a_2}\lambda}{A}\right)^{1/2}\left(\frac{\sqrt{a_2}|k_{1,2}|}{\lambda}\right)^{1/2}. \tag{108}$$

Substituting Eqs. (96) and (107) into Eq. (89), we obtain

$$c_{1,2}=B_{1,2}u_0\frac{\sqrt{a_2}|k_{1,2}|}{\lambda}\exp\left[-\frac{\pi\sqrt{a_2}|k_{1,2}|}{2\lambda}\right]. \tag{109}$$

Here we introduced complex constants $B_{1,2}$, emerging because of approximations made in determining $z_{1,2}$. These approximations lead to the errors of order one in the exponent. By the order of magnitude, $|B_{1,2}|$ ranges from one to ten. A precise calculation of $B_{1,2}$ is rather tedious. Because they are not very essential, we will not explain this in the present paper. The validity condition of expression (109) follows from Eq. (101); after simple transformations it can be written as

$$\frac{2A\lambda}{\sqrt{a_2}q^2}\ll\frac{2\lambda}{\sqrt{a_2}|k_{1,2}|}\ll1. \tag{110}$$

This is possible only if p is sufficiently close to $+1$. At $q\approx1$, from Eq. (103) we obtain $u_0\approx\lambda$ and expression (109) becomes the result obtained earlier for $\alpha_{1,2}=0$ [2].

In case (104), which can be written as

$$\frac{\lambda}{\sqrt{a_2}|k_{1,2}|}\ll\sqrt{1-p^2}=\frac{2\tilde{u}_0\tilde{u}_1}{\tilde{u}_0^2+\tilde{u}_1^2}, \tag{111}$$

we have

$$c_{1,2}=B_{1,2}\left(\frac{\sqrt{a_2}|k_{1,2}|}{A}\right)^{1/2}\times\exp\left[-\frac{\pi\sqrt{a_2}|k_{1,2}|}{4\lambda}\left(1+\frac{2}{\pi}\arcsin p\right)\right]. \tag{112}$$

The constants $B_{1,2}$ in Eq. (112) are introduced for the same reason as in Eq. (109) and they are of the same order.

Comparing Eqs. (109) and (112), we observe that at

$$\frac{\lambda A}{q^2}\sim\frac{\lambda}{|k_{1,2}|}\ll1 \tag{113}$$

(which is possible only at small $1-p$), they are of the same order. The results of Ref. [2] are still valid at condition (113); this means that, in fact, the cubic dispersive terms in

Eq. (1) are negligible (at $q\sim 1$) not only at condition (110) but in the broader region (113).

Equations (109) and (112), together with Eq. (77), describe the RSR at initial condition (70) with $t_0=-\infty$. Using these expressions for $c_{1,2}$, we can also write an asymptotic expression of the RSR for initial condition (69) at large t and $|x|$ (where the transient effects can be ignored). It reads

$$f(x,t)\cong\sum_{j=1}^{2} c_j\Theta(|U_j|t-|x|)\Theta(U_jx)e^{ik_jx}. \tag{114}$$

Here, as before, U_J are the group velocities (60). Equation (114), by comparison with Eq. (77), contains additional factors $\Theta(|U_j|t-|x|)$ which mean that the corresponding mode of the RSR at the moment t occupies the region between $x=0$ and $|x|=|U_j|t$. Due to the asymptotic character of expression (114), the soliton size as well as the fine structures of the radiation fronts are ignored.

In the limit case (63), the resonant wave numbers are approximately given by Eq. (64) and so the contribution from the second mode of the RSR can be neglected. Therefore, in this case

$$f(x,t)\cong c_1\Theta(|U_1|t-|x|)\Theta(U_1x)e^{ik_1x}, \tag{115}$$

where k_1 and U_1 are independent on a_4. From this it follows that at condition (63) one can neglect the term with the fourth derivative in Eq. (1). In this case, the RSR is emitted only in one direction, defined by the sign of U_1 (which is opposite to sgn α_3). At condition (66), Eq. (114) contains two terms with approximately equal amplitudes. From Eqs. (67) and (68) it then follows that in this case the RSR is emitted in opposite directions and the contribution from the third derivative term is negligible. Note that the second term in Eq. (114) can be neglected not only in the extreme case $|k_1|\ll|k_2|$ but also at $|k_1|<|k_2|$ (if $|k_1|$ and $|k_2|$ are not too close to each other).

VI. DISCUSSION

First, we would like to add two comments to the obtained results.

(i) Equation (A1) of the Appendix, after substitution expression (27), contains a quintic term with respect to soliton amplitude u. The term of the same order emerges if one adds the quintic term $\beta|\Psi|^4\Psi$ to Eq. (1), which can be considered as an account of the next order expansion of nonlinearity. Repeating the calculations described in the Appendix, one then arrives at the same soliton Eqs. (26) and (27) but with the modified parameter A^2:

$$A^2=\frac{4\alpha_1(\alpha_1+2\alpha_2)-(\alpha_1+2\alpha_2)^2+8\beta}{12}. \tag{116}$$

However, the new term $\beta|\Psi|^4\Psi$ in Eq. (1) leads to an appropriate addition, namely $-(\beta/3)(\Psi^*\Psi)^3$, in the Hamiltonian (7); from this it is seen that the character of the dynamics of nonlinear patterns described by modified Eq. (1) may be qualitatively changed. At sufficiently small negative β the changes are not too serious, but at $\beta>0$ the quintic term results in the soliton instability (in particular, it may lead to the collapse) [15]. In this paper we restrict ourselves with the cubic Eq. (1), which is a good model, adequately describing many phenomena in the nonlinear fiber optics. The quintic term at sufficiently small β can be ignored when the effects initiated by it are either negligible or (in the case of instability) slow enough compared with those contributed by the RSR. A detailed investigation of the systems described in Eq. (1) with the additional quintic term is given in Ref. [15].

(ii) As we have already mentioned, the soliton radiation takes energy and momentum from the soliton and this must lead to soliton modification that was neglected in the used above adiabatic approximation. The changes of soliton parameters can be evaluated analytically from the conservation laws (3), (5), and (6) under some simplifying assumptions, as was done in Ref. [2] for $\alpha_1=\alpha_2=0$, $\alpha_4=0$. In this case the soliton amplitude decreases with time logarithmically. On the contrary, the soliton velocity $V(t)$ increases (also logarithmically) and

$$\operatorname{sgn}\dot{V}(t)=-\operatorname{sgn}k_1=\operatorname{sgn}U_1. \tag{117}$$

These results were confirmed numerically [3]. An approach, extending the asymptotic method of Ref. [2] for finite but sufficiently small α_1 and $\alpha_2=0$, $\alpha_4=0$ gives the same results. A more detailed and general investigation of the evolution of radiating solitons will be done in another paper with Shagalov and Rasmussen.

In conclusion, we have studied Eq. (1), representing adequate models for important nonlinear systems. At sufficiently small $\alpha_{3,4}$, Eq. (1) describes quasisteady solitons resonantly emitting radiation. The soliton core is mainly described by Eq. (19) without the two last terms while the latter plays a decisive role in the RSR. Asymptotic expressions describing RSR are derived. The Galilei transformation, conservation laws, and Hamiltonian structures for Eq. (1) are also studied.

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APPENDIX: DERIVATION OF EQS. (26) AND (27)

Substituting Eq. (25) into Eq. (19) with $\alpha_3=\alpha_4=0$, $a_2=1$ and separating real and imaginary parts we, respectively, have

$$u''-\lambda^2u-(\varphi')^2u+2qu^3-2\alpha_1\varphi'u^3=0, \tag{A1}$$

$$\varphi''u+2\varphi'u'+2(\alpha_1+2\alpha_2)u^2u'=0. \tag{A2}$$

Multiplying Eq. (A2) by u and integrating, we arrive at Eq. (27). Substituting Eq. (27) into Eq. (A1) and then multiplying by u we have, after integration, Eq. (26).

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